Nonradiating Electromagnetic Sources in a Nonuniform Medium

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Abstract

Nonradiating electromagnetic sources are sources whose field is identically zero outside of their volume. They are undetectable unless the observation point is in direct contact with them. They are the basis of the theory of source equivalence, which studies the field invariance with respect to source transformations. In this work, we focus on the equivalent source transformations in a nonuniform medium and the implications in the theory of the electromagnetic vector potentials. We identify three types of nonradiating sources. Subsequently, we define the mathematical transformations of the sources, which preserve the field outside of their support (source invariance). We give complimentary expressions, which preserve the field inside the source support as well. We show that the nonuniqueness of the electromagnetic potentials is due to the nonunique solution to the inverse problem. The well known field gauge invariance follows from its source invariance. Also, the gauge–invariant transformation appears to be just one possibility in an infinite set of field–invariant vector–potential representations all related to the respective equivalent source transformations.

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I. INTRODUCTION AND DEFINITIONS

The sources of the electromagnetic field are time–varying electric and magnetic current densities, the latter being commonly treated as fictitious quantities. We assume that the charge densities are related to their respective current densities via the continuity law.

Two sources are considered *equivalent* if they generate identical fields. In general, the fields have to be identical outside the volumes of each of the sources and their spatial derivatives, while they may differ inside [1]. Hereafter, the volume where a source and its derivatives are nonzero is referred to as its *support*. We assume that this support is finite.

A source generating an electromagnetic field confined within its support is called *nonradiating* [2]. A nonradiating source is undetectable unless the observation point is on or within its volume. It is now obvious that two sources are equivalent if their difference is a nonradiating source of bound support [1]. Their fields are equivalent outside of the support of the nonradiating "difference" source. Thus, the source equivalence requires a clear understanding of the nonradiating electromagnetic sources.

Nonradiating sources and source equivalence have been considered almost exclusively in a uniform medium, see, for example, [1–6]. Classes of nonradiating sources have been identified in optics, see for example [7, 8] based on the equation of radiative transfer as well as its diffusion approximation, i.e., the equation of photon density distribution. General approaches to the identification of the radiating and nonradiating parts of a source distribution have also been proposed [9, 10]. The attention has always been on the field extinction outside of the source support, while the field within the nonradiating source is usually not considered.

Notably, the specifics of the case of a nonuniform medium have been considered in [11]—a work where the construction of equivalent sources for numerical computations is the focus. The mathematical procedure for the construction of the equivalent source is based on a Helmholtz representation of the source vectors. Arbitrarily oriented electric–magnetic source configurations are transformed into equivalent single–component co–directional ($\hat{\mathbf{n}}$-oriented) densities of electric and magnetic currents distributed in planes orthogonal to $\hat{\mathbf{n}}$. The procedure is referred to as TE/TM source decomposition, also, source scalarization. The transformation is independent of the constitutive parameters, and it allows field computations in terms of two scalar functions of space–time.

In this work, for the first time, we state clearly the mathematical form of the nonradiating sources and the related source equivalence in the case of a nonuniform isotropic medium—a case of significant importance in computational electromagnetics and practical inverse scattering problems. We also show how to recover the original field inside the source support. We thus hope to fill in a gap in the current knowledge on the nature of electromagnetic source equivalence and its numerical implementation.

Most importantly, we consider the nonradiating sources not only in the context of field–based analysis with Maxwell's equations but also in the context of the electromagnetic vector potentials. We show that the nonuniqueness of the vector–potential representation of the electromagnetic field is a direct consequence of the nonunique solution to the inverse problem. In other words, for a given physically observable field, mathematically, there exists an infinite set of sources capable of generating it. Accordingly, there is an infinite set of vector potentials, which represent the same electromagnetic field. What is considered by some to be a "drawback" of the vector potentials as opposed to the field vectors **E** and **H** is, in fact, a remarkable feature allowing us to identify sources, which are "nonradiating" for the field vectors but are "radiating" for the vector potentials. This theory has potentially important implications in the ongoing dispute on the measurability of the vector potentials as well as in the electromagnetic source identification.

The analysis is carried out in the time domain; however, the results are easily transferrable into the frequency domain as explained in the next section.

The paper is organized as follows. We first define three basic types of nonradiating sources. We then consider the equivalence of sources, and suggest a general approach to the construction of equivalent sources for a given original set of sources. Further, we consider the nonradiating sources in conjunction with the vector potentials. We focus on the related nonuniqueness of the vector potentials and the electromagnetic field invariance.

II. NOTATIONS

First–order partial derivatives are denoted concisely by ∂_ξ, i.e. $\partial_\xi = \frac{\partial}{\partial \xi}$. Similarly, higher–order derivatives appear as $\partial_{\xi\zeta} = \frac{\partial^2}{\partial\xi\partial\zeta}$, etc.

The operators $\mathcal{T}_\epsilon$ and $\mathcal{T}_\mu$ relate to the time–derivatives of the flux densities **D** and **B**, respectively,

$$\partial_t \mathbf{D} = \mathcal{T}_\epsilon \mathbf{E}\,,\; \partial_t \mathbf{B} = \mathcal{T}_\mu \mathbf{H}\,. \tag{1}$$

For example, in a nondispersive medium, $\mathcal{T}_\epsilon = \boldsymbol{\epsilon}\partial_t + \boldsymbol{\sigma}_e$ and $\mathcal{T}_\mu = \boldsymbol{\mu}\partial_t + \boldsymbol{\sigma}_m$, where $\boldsymbol{\epsilon}$ is the permittivity, $\boldsymbol{\mu}$ is the permeability, $\boldsymbol{\sigma}_e$ and $\boldsymbol{\sigma}_m$ are the specific electric and magnetic conductivities, respectively. The formulas in Eq. (1) are in effect the time–dependent constitutive relations. The involved constitutive parameters are tensors in the case of an anisotropic medium although here we focus on the isotropic case.

We also make use of the operators $\mathcal{T}_\epsilon^{-1}$ and $\mathcal{T}_\mu^{-1}$, which are the inverse of $\mathcal{T}_\epsilon$ and $\mathcal{T}_\mu$, respectively. They have simple analytical form in the case of a dispersion–free loss–free medium, e.g., $\mathcal{T}_\epsilon^{-1} = \boldsymbol{\epsilon}^{-1}\int_t$. In the case of a dispersive lossy medium, general analytical expressions are not available, however, their discrete numerical counterparts always exist. We also use the second–order operator $\mathcal{T}_{\mu\epsilon} = \mathcal{T}_\mu \mathcal{T}_\epsilon$. In a nondispersive medium, $\mathcal{T}_{\mu\epsilon} = \boldsymbol{\mu}\boldsymbol{\epsilon}\partial_{tt} + (\boldsymbol{\epsilon}\boldsymbol{\sigma}_m + \boldsymbol{\mu}\boldsymbol{\sigma}_e)\partial_t + \boldsymbol{\sigma}_e\boldsymbol{\sigma}_m$.

In the nonuniform medium analysis, where the gradients of the constitutive parameters are involved, we use the gradient vector operators $(\nabla\mathcal{T}_\epsilon)$ and $(\nabla\mathcal{T}_\mu)$:

$$\begin{aligned} (\nabla\mathcal{T}_\epsilon) &= (\nabla\epsilon)\partial_t + (\nabla\sigma_e)\,, \\ (\nabla\mathcal{T}_\mu) &= (\nabla\mu)\partial_t + (\nabla\sigma_m)\,, \end{aligned} \tag{2}$$

so that, for example,

$$\begin{aligned} &(\nabla\mathcal{T}_\epsilon)\,\Phi = (\nabla\epsilon)\partial_t\Phi + (\nabla\sigma_e)\Phi\,, \\ &(\nabla\mathcal{T}_\epsilon)\times\mathbf{F}_\epsilon = (\nabla\epsilon)\times\partial_t\mathbf{F}_\epsilon + (\nabla\sigma_e)\times\mathbf{F}_\epsilon\,, \end{aligned} \tag{3}$$

where Φ is a scalar and $\mathbf{F}_\epsilon$ is a vector.

The time–domain analysis given hereafter can be directly transferred into the frequency domain with the replacement of the operators $\mathcal{T}_\epsilon$ and $\mathcal{T}_\mu$ by $j\omega\tilde{\boldsymbol{\epsilon}}$ and $j\omega\tilde{\boldsymbol{\mu}}$, respectively, in the phasor form of Maxwell's equations, or in the Helmholtz equation. Here, $\tilde{\boldsymbol{\epsilon}}$ and $\tilde{\boldsymbol{\mu}}$ are the complex permittivity and permeability, respectively.

III. NONRADIATING CURRENT SOURCES IN A NONUNIFORM MEDIUM

Previous work, e.g. [1, 2], formulates two classes of nonradiating current densities in a uniform medium—those expressible in terms of the gradient of a scalar, and those which are functions of a vector. Here, we give expressions for nonradiating electromagnetic sources valid in nonuniform media. They appear as generalizations of the sources discussed in [1, 2]. We add a third class of nonradiating sources (see Theorem 2), which uses a specific combination of electric and magnetic current densities.

Theorem 1 *Sources of bound support in the form of electric current density*

$$\mathbf{J}_e^{nr} = \mathcal{T}_\epsilon \nabla P_e \tag{4}$$

and/or magnetic current density

$$\mathbf{J}_m^{nr} = \mathcal{T}_\mu \nabla P_m \tag{5}$$

do not generate an electromagnetic field outside of their own support. P_e *and* P_m *can be any scalar functions of space–time whose first–order derivatives are well defined. The field they generate is localized at points of nonzero* $\mathbf{J}_e^{nr}$*, where*

$$\mathbf{E} = -\nabla P_e \,, \tag{6}$$

and nonzero $\mathbf{J}_m^{nr}$*, where*

$$\mathbf{H} = -\nabla P_m \,. \tag{7}$$

Proof:

The field excited by $\mathbf{J}_e^{nr}$ and $\mathbf{J}_m^{nr}$ satisfies the Maxwell equations

$$\begin{aligned} \nabla \times \mathbf{H} &= \mathcal{T}_\epsilon \mathbf{E} + \mathcal{T}_\epsilon \nabla P_e \,, \\ -\nabla \times \mathbf{E} &= \mathcal{T}_\mu \mathbf{H} + \mathcal{T}_\mu \nabla P_m \,. \end{aligned} \tag{8}$$

We are interested in the particular solution corresponding to the specified sources only, and to the field zero boundary conditions. Eq. (8) can be re-written as

$$\begin{aligned} \nabla \times \mathbf{H}' &= \mathcal{T}_\epsilon \mathbf{E}' \,, \\ -\nabla \times \mathbf{E}' &= \mathcal{T}_\mu \mathbf{H}' \,, \end{aligned} \tag{9}$$

where $\mathbf{H}' = \mathbf{H} + \nabla P_m$ and $\mathbf{E}' = \mathbf{E} + \nabla P_e$. Since $\mathbf{J}_e^{nr}$ and $\mathbf{J}_m^{nr}$ are of bound support, the particular solution of interest to Eq. (9) is trivial: $\mathbf{H}' = \mathbf{0}$, $\mathbf{E}' = \mathbf{0}$. Thus, the field $\mathbf{E}$ and $\mathbf{H}$ generated by $\mathbf{J}_e^{nr}$ and $\mathbf{J}_m^{nr}$ is localized at the sources and is given by Eq. (6) and Eq. (7). ■

A simple example is the current density in a conducting medium under steady–current conditions, where $\mathbf{E} = -\nabla\phi$, ϕ being the electric scalar potential. The current density is $\mathbf{J}_e = -\sigma_e \nabla\phi$. This is a particular form of Eq. (4) where $\phi = P_e$, and $\mathcal{T}_\epsilon$ reduces to $\mathcal{T}_\epsilon = \sigma_e$ since ϕ is constant in time, $\epsilon\partial_t\phi = 0$. It is well known that steady currents do not radiate.

However, in a uniform medium, even if the current is time–varying, it would not radiate beyond its support, provided that in space it can be represented by the gradient of a scalar: $\mathbf{J}_e^{nr} = \nabla(\mathcal{T}_\epsilon P_e)$. Curl–free currents do not radiate. They are due to external electromotive forces, which are conservative in nature. Gradient type nonradiating sources in a uniform medium are discussed in [1], see also [9]. Note that the source equal to the gradient of a scalar may, in general, radiate if it exists in a locally nonuniform medium.

Theorem 2 *Combinations of electric and magnetic current densities of the forms*

$$\mathbf{J}_e^{nr} = -\mathcal{T}_\epsilon \boldsymbol{\mathcal{E}},\ \mathbf{J}_m^{nr} = -\nabla \times \boldsymbol{\mathcal{E}}\,, \tag{10}$$

and

$$\mathbf{J}_m^{nr} = -\mathcal{T}_\mu \boldsymbol{\mathcal{H}},\ \mathbf{J}_e^{nr} = \nabla \times \boldsymbol{\mathcal{H}}\,, \tag{11}$$

where $\boldsymbol{\mathcal{E}}$ *and* $\boldsymbol{\mathcal{H}}$ *can be any vectors with well defined derivatives, produce zero field outside of their own support.*

Proof:

Consider the source set in Eq. (10). Substituting it in Maxwell's equations, we have

$$\begin{aligned} \nabla \times \mathbf{H} &= \mathcal{T}_\epsilon(\mathbf{E} - \boldsymbol{\mathcal{E}})\,, \\ -\nabla \times (\mathbf{E} - \boldsymbol{\mathcal{E}}) &= \mathcal{T}_\mu \mathbf{H}\,. \end{aligned} \tag{12}$$

As before, we are concerned only with the particular solution due to the specified sources (with zero boundary conditions for the field). With respect to the vectors $\mathbf{H}$ and $\mathbf{E}' = \mathbf{E} - \boldsymbol{\mathcal{E}}$, the system above is homogeneous, i.e., source–free; therefore, the solution is

$$\mathbf{E} = \boldsymbol{\mathcal{E}},\ \mathbf{H} = \mathbf{0}\,. \tag{13}$$

Analogously, in the case of the source given by Eq. (11), the solution appears as

$$\mathbf{E} = \mathbf{0},\ \mathbf{H} = \boldsymbol{\mathcal{H}}\,. \tag{14}$$

Both solutions are local in the sense that they are nonzero only inside the source. ■

As an example, consider the time–varying current in a wire loop whose density is expressed as $\mathbf{J}_e = \hat{\boldsymbol{\varphi}}\epsilon\partial_t\mathcal{E}$, where $\hat{\boldsymbol{\varphi}}$ is the unit vector in cylindrical coordinates, and $\mathcal{E}$ is constant along the loop. The loop is centered onto the origin of the coordinate system and its axis is along the z-axis. Its radius is a. A magnetic current I_m along the z-axis can now be defined:

$$I_m = \iint\limits_A \mathbf{J}_m \cdot \hat{\boldsymbol{z}} ds\,. \tag{15}$$

We set $\mathbf{J}_m = \nabla \times \mathcal{E}$ as per Eq. (10). Using Stokes theorem, we find that

$$I_m = 2\pi a\mathcal{E}\,, \mathrm{V}\,. \tag{16}$$

If I_m is placed at the origin together with the electric–current loop, their respective fields will cancel outside of the volume of the loop, resulting in a zero total field. The equivalence of the fields due to small electrical loops and magnetic dipoles has long been known to antenna engineers, and is widely used in conjunction with electromagnetic duality, see, for example, [12].

Theorem 3 *Assume that the vector field* $\mathbf{A}$ *satisfies the equation*

$$L\mathbf{A} = \mathbf{G}\,, \tag{17}$$

where L *is any linear operator, and* $\mathbf{G}$ *represents sources. If sources exist, which are expressible in the from*

$$\mathbf{G}^{nr} = L\mathbf{B}\,, \tag{18}$$

then these sources do not radiate beyond their own support. The vector $\mathbf{B}$ *is exactly the nonpropagating vector field generated by* $\mathbf{G}^{nr}$*.*

Proof:

In the presence of the source in Eq. (18), Eq. (17) can be written as

$$L(\mathbf{A} - \mathbf{B}) = \mathbf{0}\,. \tag{19}$$

As both vector fields $\mathbf{A}$ and $\mathbf{B}$ satisfy the same zero boundary conditions, the above equation has only a trivial particular solution for the given sources. Thus, $\mathbf{A} \equiv \mathbf{B}$. ■

The above theorem leads to the formulation of yet another type of nonradiating electromagnetic sources. Having in mind the vector equation for the **E**-field in the presence of electrical current densities,

$$\nabla \times \mathcal{T}_\mu^{-1} \nabla \times \mathbf{E}_e + \mathcal{T}_\epsilon \mathbf{E}_e = -\mathbf{J}_e \,, \tag{20}$$

we conclude that in a nonuniform medium electrical sources of the type

$$-\mathbf{J}_e^{nr} = \nabla \times \mathcal{T}_\mu^{-1} \nabla \times \boldsymbol{\mathcal{E}} + \mathcal{T}_\epsilon \boldsymbol{\mathcal{E}} \tag{21}$$

produce zero **E**-field outside of the support of $\boldsymbol{\mathcal{E}}$, i.e., $\mathbf{E}_e^{nr} = \boldsymbol{\mathcal{E}}$. Then, from Maxwell's equations, $\mathbf{H}_e^{nr} = -\mathcal{T}_\mu^{-1} \nabla \times \boldsymbol{\mathcal{E}}$.

Similarly, the vector equation for the **H**-field in the presence of magnetic sources is

$$\nabla \times \mathcal{T}_\epsilon^{-1} \nabla \times \mathbf{H}_m + \mathcal{T}_\mu \mathbf{H}_m = -\mathbf{J}_m \,. \tag{22}$$

Thus, if these sources are given by

$$-\mathbf{J}_m^{nr} = \nabla \times \mathcal{T}_\epsilon^{-1} \nabla \times \boldsymbol{\mathcal{H}} + \mathcal{T}_\mu \boldsymbol{\mathcal{H}} \,, \tag{23}$$

then $\mathbf{H}_m^{nr} = \boldsymbol{\mathcal{H}}$, and $\mathbf{E}_m^{nr} = \mathcal{T}_\epsilon^{-1} \nabla \times \boldsymbol{\mathcal{H}}$.

The nonradiating sources defined by Eq. (21) and Eq. (23) are generalizations of a nonradiating source first considered by Devaney *et al.* [2].

IV. EQUIVALENCE OF CURRENT SOURCES

A. Equivalence Between Electric and Magnetic Current Sources

Theorem 4 *The field generated by the magnetic current density* $\mathbf{J}_m$ *is identical to the field generated by the electric current density* $\mathbf{J}_e$ *outside the sources' support provided that*

$$\nabla \times \mathcal{T}_\mu^{-1} \mathbf{J}_m = \mathbf{J}_e \,, \tag{24}$$

or

$$-\nabla \times \mathcal{T}_\epsilon^{-1} \mathbf{J}_e = \mathbf{J}_m \,. \tag{25}$$

Proof:

This theorem follows directly from Theorem 2. For example, assume that $\mathbf{J}_m$ in Eq. (24) is expressed as $\mathbf{J}_m = -\mathcal{T}_\mu \boldsymbol{\mathcal{H}}$. It generates the field $(\mathbf{E}_m, \mathbf{H}_m)$. It then follows that $\mathbf{J}_e$ in Eq. (24) is $\mathbf{J}_e = -\nabla \times \boldsymbol{\mathcal{H}}$. Let its field be $(\mathbf{E}_e, \mathbf{H}_e)$. If we consider the source set formed by $\mathbf{J}_m$ and $-\mathbf{J}_e$, we see that it is nonradiating, see Eq. (11). This proves that $\mathbf{J}_e$ and $\mathbf{J}_m$ are equivalent, and their respective fields are identical outside of the support of $\boldsymbol{\mathcal{H}}$. The difference field is nonzero only inside the support of $\boldsymbol{\mathcal{H}}$, and it is

$$\begin{aligned} \mathbf{E}_m - \mathbf{E}_e &= \mathbf{0}\,, \\ \mathbf{H}_m - \mathbf{H}_e &= \boldsymbol{\mathcal{H}}\,. \end{aligned} \tag{26}$$

Analogous proof holds for the equivalence of the sources given in Eq. (25). This time, the difference between the two fields is

$$\begin{aligned} \mathbf{E}_e - \mathbf{E}_m &= \boldsymbol{\mathcal{E}}\,, \\ \mathbf{H}_e - \mathbf{H}_m &= \mathbf{0}\,, \end{aligned} \tag{27}$$

where $(\mathbf{E}_e, \mathbf{H}_e)$ is the field due to $\mathbf{J}_e = -\mathcal{T}_\epsilon \boldsymbol{\mathcal{E}}$, and $(\mathbf{E}_m, \mathbf{H}_m)$ is the field due to $\mathbf{J}_m = \nabla \times \boldsymbol{\mathcal{E}}$. ■

To recapitulate, we re-express Theorem 4 in terms of the auxiliary vectors $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{H}}$, and state that *the electric current density* $\mathbf{J}_e = \nabla \times \boldsymbol{\mathcal{H}}$ *is equivalent to the magnetic current density* $\mathbf{J}_m = \mathcal{T}_\mu \boldsymbol{\mathcal{H}}$. *The dual equivalence involves* $\mathbf{J}_m = \nabla \times \boldsymbol{\mathcal{E}}$ *and* $\mathbf{J}_e = -\mathcal{T}_\epsilon \boldsymbol{\mathcal{E}}$.

Theorem 4 is an extension of the electric–magnetic source equivalence considered first by Mayes [4] to the case of a nonuniform medium.

The above theorem suggests the existence of two equivalent current distributions for a given original source. This deserves some additional comment. Suppose the original source is given by $\mathbf{J}_e^o$. Then, there are two possible equivalent magnetic current distributions, $\mathbf{J}_{m1}^e$ and $\mathbf{J}_{m2}^e$, such that

$$\nabla \times \mathcal{T}_\mu^{-1} \mathbf{J}_{m1}^e = \mathbf{J}_e^o\,, \tag{28}$$

$$\mathbf{J}_{m2}^e = -\nabla \times \mathcal{T}_\epsilon^{-1} \mathbf{J}_e^o\,. \tag{29}$$

Hence, $\mathbf{J}_{m1}^e$ and $\mathbf{J}_{m2}^e$ must be mutually equivalent, too. This is true, and it can be shown by considering their difference. We apply the operator $\nabla \times \mathcal{T}_\mu^{-1}$ to Eq. (29) and form an expression for the difference of the two magnetic equivalent sources:

$$\nabla \times \mathcal{T}_\mu^{-1} (\mathbf{J}_{m2}^e - \mathbf{J}_{m1}^e) = -(\nabla \times \mathcal{T}_\mu^{-1} \nabla \times \mathcal{T}_\epsilon^{-1} \mathbf{J}_e^o + \mathbf{J}_e^o)\,. \tag{30}$$

The left–hand side, by its dimensionality, corresponds to an equivalent electric current density (see also Eq. (24)), $\delta\mathbf{J}_e^e$, such that

$$\delta\mathbf{J}_e^e = -(\nabla \times \mathcal{T}_\mu^{-1}\nabla \times \mathcal{T}_\epsilon^{-1}\mathbf{J}_e^o + \mathbf{J}_e^o)\,, \tag{31}$$

which can be re-expressed in the form

$$\delta\mathbf{J}_e^e = -(\nabla \times \mathcal{T}_\mu^{-1}\nabla \times \boldsymbol{\mathcal{E}} + \mathcal{T}_\epsilon\boldsymbol{\mathcal{E}})\,, \tag{32}$$

where $\mathbf{J}_e^o = \mathcal{T}_\epsilon\boldsymbol{\mathcal{E}}$. This expression, as we showed before in Eq. (21), indicates a nonradiating electric source in a nonuniform medium.

The equivalence of any two sources can be validated by considering their difference. If it produces a nonradiating source, the fields of these two sources are identical outside of the support of this "difference" source.

B. Recursive Source Transformations

Provided that the higher–order derivatives of the original current densities exist, we can apply the equivalent source transformations Eq. (24) and Eq. (25) recursively. In other words, we can find the equivalent of the original source, then the equivalent of the equivalent, and so on. Therefore, there is an infinite number of equivalent sources for an original source whose higher–order derivatives exist.

Consider as an example an original source given by $\mathbf{J}_{e1} = -\mathcal{T}_\epsilon\boldsymbol{\mathcal{E}}_1$. From Eq. (25), an equivalent magnetic current density $\mathbf{J}_{m2} = \nabla \times \boldsymbol{\mathcal{E}}_1$ is found, with the equivalent fields being $\mathbf{H}_2 = \mathbf{H}_1$, $\mathbf{E}_2 = \mathbf{E}_1 - \boldsymbol{\mathcal{E}}_1$, as per Eq. (27). Further, from Eq. (24), we find the equivalent $\mathbf{J}_{e3}$ as

$$\mathbf{J}_{e3} = \nabla \times \mathcal{T}_\mu^{-1}\mathbf{J}_{m2} = \nabla \times \mathcal{T}_\mu^{-1}\nabla \times \boldsymbol{\mathcal{E}}_1\,. \tag{33}$$

This can be written also as

$$\mathcal{T}_\epsilon\boldsymbol{\mathcal{E}}_3 = -\nabla \times \mathcal{T}_\mu^{-1}\nabla \times \boldsymbol{\mathcal{E}}_1\,, \tag{34}$$

where $\mathbf{J}_{e3} = -\mathcal{T}_\epsilon\boldsymbol{\mathcal{E}}_3$. The field $(\mathbf{E}_3, \mathbf{H}_3)$, generated by $\mathbf{J}_{e3}$ relates to $(\mathbf{E}_2, \mathbf{H}_2)$ as $\mathbf{E}_3 = \mathbf{E}_2$, $\mathbf{H}_3 = \mathbf{H}_2 + \mathcal{T}_\mu^{-1}\nabla \times \boldsymbol{\mathcal{E}}_1$, according to Eq. (26), where we have substituted $\boldsymbol{\mathcal{H}}_2 = -\mathcal{T}_\mu^{-1}\nabla \times \boldsymbol{\mathcal{E}}_1$. The latter comes from representing $\mathbf{J}_{m2}$ as $\mathbf{J}_{m2} = -\mathcal{T}_\mu\boldsymbol{\mathcal{H}}_2$ as in the proof of Theorem 4, and then equating this to $\mathbf{J}_{m2} = \nabla \times \boldsymbol{\mathcal{E}}_1$. Finally, the difference between the equivalent field $(\mathbf{E}_3, \mathbf{H}_3)$ and the original field $(\mathbf{E}_1, \mathbf{H}_1)$, is

$$\begin{aligned} \mathbf{H}_3 - \mathbf{H}_1 &= \mathcal{T}_\mu^{-1} \nabla \times \boldsymbol{\mathcal{E}}_1 \,, \\ \mathbf{E}_3 - \mathbf{E}_1 &= -\boldsymbol{\mathcal{E}}_1 \,. \end{aligned} \tag{35}$$

The difference field is nonzero only within the support of $\boldsymbol{\mathcal{E}}_1$.

Note that the equivalence of the first and the third source is evident also from the fact that their difference yields

$$\mathbf{J}_{e3} - \mathbf{J}_{e1} = \mathcal{T}_\epsilon \boldsymbol{\mathcal{E}}_1 + \nabla \times \mathcal{T}_\mu^{-1} \nabla \times \boldsymbol{\mathcal{E}}_1 \,, \tag{36}$$

which is a nonradiating source of the form of Eq. (21) with a minus sign. The difference field derived in Eq. (35) corresponds exactly to the field of the nonradiating source defined by Eq. (21) with a minus sign.

In the time domain, the second–order source transformation given by Eq. (34) leads to a sequence of sources derived from each other as time progresses. This equivalent–source "propagation" is wave–like and very much analogous to the way the field itself propagates.

Higher–order recursive transformations are possible with sources whose respective derivatives exist.

V. NONRADIATING SOURCES AND VECTOR POTENTIALS

A. Vector potentials in a nonuniform medium

The vector–potential representation in a nonuniform isotropic medium in the presence of electrical and magnetic currents is [13, 14]:

$$\nabla^2 \mathbf{A}_\mu - \mathcal{T}_{\mu\epsilon} \mathbf{A}_\mu + (\nabla \mathcal{T}_\epsilon) \times \mathbf{F}_\epsilon + (\nabla \mathcal{T}_\epsilon) \mathit{\Phi} = -\mathbf{J}_e \,, \tag{37}$$

$$\nabla^2 \mathbf{F}_\epsilon - \mathcal{T}_{\mu\epsilon} \mathbf{F}_\epsilon - (\nabla \mathcal{T}_\mu) \times \mathbf{A}_\mu + (\nabla \mathcal{T}_\mu) \mathit{\Psi} = -\mathbf{J}_m \,, \tag{38}$$

where the magnetic potential $\mathbf{A}_\mu$ (measured in *amperes*) and the electric potential $\mathbf{F}_\epsilon$ (measured in *volts*) relate to the conventional vector potentials $\mathbf{A}$ and $\mathbf{F}$ as $\mathbf{A}_\mu = \mathbf{A}/\mu$ and $\mathbf{F}_\epsilon = \mathbf{F}/\epsilon$. The scalar potentials $\mathit{\Phi}$ and $\mathit{\Psi}$ are defined through a generalized Lorenz gauge

$$\begin{aligned} -\mathcal{T}_\epsilon \mathit{\Phi} &= \nabla \cdot \mathbf{A}_\mu \,, \\ -\mathcal{T}_\mu \mathit{\Psi} &= \nabla \cdot \mathbf{F}_\epsilon \,. \end{aligned} \tag{39}$$

The vector–potential equations Eqs. (37)–(38) are coupled if the medium is nonuniform. Otherwise, they decouple and reduce to the conventional wave equations:

$$\nabla^2 \mathbf{A}_\mu - \mathcal{T}_{\mu\epsilon} \mathbf{A}_\mu = -\mathbf{J}_e \,, \tag{40}$$

$$\nabla^2 \mathbf{F}_\epsilon - \mathcal{T}_{\mu\epsilon} \mathbf{F}_\epsilon = -\mathbf{J}_m \,. \tag{41}$$

The field–potential relations are obtained by writing Eqs. (37)–(38) in a form equivalent to that of Maxwell's equations [13]:

$$\begin{aligned} -\nabla \times (-\mathcal{T}_\epsilon \mathbf{F}_\epsilon - \nabla \mathit{\Psi} + \nabla \times \mathbf{A}_\mu) + \\ \mathcal{T}_\epsilon(-\mathcal{T}_\mu \mathbf{A}_\mu - \nabla \mathit{\Phi} - \nabla \times \mathbf{F}_\epsilon) = -\mathbf{J}_e \,, \end{aligned} \tag{42}$$

$$\begin{aligned} \nabla \times (-\mathcal{T}_\mu \mathbf{A}_\mu - \nabla \mathit{\Phi} - \nabla \times \mathbf{F}_\epsilon) + \\ \mathcal{T}_\mu(-\mathcal{T}_\epsilon \mathbf{F}_\epsilon - \nabla \mathit{\Psi} + \nabla \times \mathbf{A}_\mu) = -\mathbf{J}_m \,. \end{aligned} \tag{43}$$

The comparison between Eqs. (42)–(43) and Maxwell's equations,

$$\begin{aligned} -\nabla \times \mathbf{H} + \mathcal{T}_\epsilon \mathbf{E} = -\mathbf{J}_e \,, \\ \nabla \times \mathbf{E} + \mathcal{T}_\mu \mathbf{H} = -\mathbf{J}_m \,, \end{aligned} \tag{44}$$

shows that the field vectors appear in terms of the vector potentials as

$$\begin{aligned} \mathbf{E} = -\mathcal{T}_\mu \mathbf{A}_\mu - \nabla \mathit{\Phi} - \nabla \times \mathbf{F}_\epsilon \,, \\ \mathbf{H} = -\mathcal{T}_\epsilon \mathbf{F}_\epsilon - \nabla \mathit{\Psi} + \nabla \times \mathbf{A}_\mu \,. \end{aligned} \tag{45}$$

The substitution of Eqs. (45) into Maxwell's system leads to an alternative field–potential relation:

$$\begin{aligned} \mathcal{T}_\epsilon \mathbf{E} = \nabla \times (\nabla \times \mathbf{A}_\mu - \mathcal{T}_\epsilon \mathbf{F}_\epsilon) - \mathbf{J}_e \,, \\ \mathcal{T}_\mu \mathbf{H} = \nabla \times (\nabla \times \mathbf{F}_\epsilon + \mathcal{T}_\mu \mathbf{A}_\mu) - \mathbf{J}_m \,. \end{aligned} \tag{46}$$

Eq. (45) and Eq. (46) are equivalent.

B. Nonradiating gradient sources and gauge invariance

Assume the existence of nonradiating sources of the type given by Eq. (4)–(5). They generate vector potentials, $\mathbf{A}_\mu^{nr}$ and $\mathbf{F}_\epsilon^{nr}$, whose solutions, as shown in Appendix B, are

$$\mathcal{T}_\mu \mathbf{A}_\mu^{nr} = \nabla \Lambda_\mu \,; \; \mathcal{T}_\epsilon \mathbf{F}_\epsilon^{nr} = \nabla \Lambda_\epsilon \,. \tag{47}$$

Here, $\Lambda_\mu = P_e - \varphi$ and $\Lambda_\epsilon = P_m - \psi$, with φ and ψ being the scalar potentials to $\mathbf{A}_\mu^{nr}$ and $\mathbf{F}_\epsilon^{nr}$, respectively. Since $\mathbf{A}_\mu^{nr}$ and φ, as well as $\mathbf{F}_\epsilon^{nr}$ and ψ, relate through the Lorenz gauge Eq. (39), it follows that both Λ_μ and Λ_ϵ satisfy wave–like equations, e.g.,

$$\mathcal{T}_\mu \nabla \cdot \mathcal{T}_\mu^{-1} \nabla \Lambda_\mu - \mathcal{T}_{\mu\epsilon} \Lambda_\mu = -\mathcal{T}_{\mu\epsilon} P_e \,. \tag{48}$$

The equation for Λ_ϵ is dual. Outside the support of the nonradiating sources, these equations are homogeneous (source–free).

Also, in Appendix A, we show that the vector–potential model is in accordance with Theorem 1: it yields only locally nonzero fields, $\mathbf{E} = -\nabla P_e$, $\mathbf{H} = -\nabla P_m$, in the case of gradient–type nonradiating sources, $\mathbf{J}_e^{nr} = \mathcal{T}_\epsilon P_e$, $\mathbf{J}_m^{nr} = \mathcal{T}_\mu P_m$.

The above discussion is closely related to the field gauge invariance. To make it more transparent, we consider the case of a uniform loss–free medium, which is customary for the classical vector potential theory. We first summarize the well–known electromagnetic field and gauge invariance. Adding a gradient term to the magnetic vector potential $\mathbf{A}$ ($\mathbf{A} = \mu \mathbf{A}_\mu$),

$$\mathbf{A}' = \mathbf{A} + \boldsymbol{\mathcal{A}} \,, \; \boldsymbol{\mathcal{A}} = \nabla \Lambda \,, \tag{49}$$

changes neither the $\mathbf{E}$ nor the $\mathbf{H}$ field vector provided that, in the same time, the electric scalar potential Φ changes as

$$\Phi' = \Phi + \varphi \,, \; \varphi = -\partial_t \Lambda \,. \tag{50}$$

Since both the original 4-vector $(\mathbf{A}, \Phi)$ and the transformed one $(\mathbf{A}', \Phi')$ must satisfy the same (Lorenz) gauge, the scalar function Λ is not completely arbitrary: it must satisfy the homogeneous wave equation,

$$\nabla^2 \Lambda - \mu\epsilon \partial_{tt} \Lambda = 0 \,. \tag{51}$$

We now turn back to the vector potential $\mathbf{A}_\mu^{nr}$, see Eq. (47), due to a gradient–type nonradiating current $\mathbf{J}_e^{nr} = \mathcal{T}_\epsilon \nabla P_e$ in the case of a uniform loss–free medium, and compare with the field gauge invariance. Eq. (47) can be written as

$$\partial_t \boldsymbol{\mathcal{A}} = \nabla \Lambda_\mu \,, \tag{52}$$

where $\boldsymbol{\mathcal{A}} = \mu \mathbf{A}_\mu^{nr}$. Comparing Eq. (52) with Eq. (49), we relate Λ_μ and Λ as $\Lambda_\mu = \partial_t \Lambda$, which makes both equations identical. Also,

$$\partial_t \Lambda = P_e - \varphi \,. \tag{53}$$

This is identical with the second equality in Eq. (50) when $P_e = 0$, i.e., outside the support of the nonradiating current density $\mathbf{J}_e^{nr}$.

We also note that, as follows from Eq. (48),

$$\nabla^2 \Lambda - \mu\epsilon \partial_{tt} \Lambda = -\mu\epsilon \partial_t P_e \, , \tag{54}$$

which is identical with Eq. (51) outside the nonradiating source.

We complete this comparison by noting that the nonradiating source $\mathbf{J}_e^{nr} = \mathcal{T}_\epsilon P_e$ leads to an $\mathbf{H}$ field vector, which is identically zero everywhere, and an $\mathbf{E}$ field vector, which is nonzero only locally within the source support, $\mathbf{E} = -\nabla P_e$.

The above comparison can be repeated with regard to the electric vector potential $\mathbf{F}$.

To summarize, adding a gradient–type nonradiating source to an original set of sources, leads to a change in the vector potential in the form of the gradient of a scalar function Λ, which satisfies the wave equation. The field vectors remain unchanged except within the support of the nonradiating source. All changes occurring with the vector and scalar potentials outside the nonradiating source are identical with those associated with a gauge–invariant transformation. The gauge–associated nonuniqueness of the vector potentials is due to the nonuniqueness of the solution to the inverse electromagnetic problem, and, in particular, to the field invariance with regard to gradient–type additions to its sources.

We also emphasize that a gradient–type nonradiating source for the $\mathbf{E}$ and $\mathbf{H}$ field vectors is in fact a "radiating" source for the vector potential. This is a situation, which is desirable in the experiments on the measurability of the electromagnetic potentials. The mathematical reason for this difference lies in the different linear operators applied to the pair $(\mathbf{E}, \mathbf{H})$ in the Maxwell equations and the vector potential in the wave equation. It is discussed in more detail at the end of this section.

C. Vector potentials and the equivalence between electric and magnetic current sources

There is more to the nonuniqueness of the vector potential representation of the electromagnetic field than the gradient term considered above.

It is customary to think of the electrical current densities as the sources of the magnetic vector potential, and of the magnetic current densities as the sources of the electrical vector potential, see Eqs. (37)–(38). The equivalence between electric and magnetic currents given by Theorem 4 suggests that there exists a similar equivalence between the magnetic and electric vector potentials.

We limit the analysis to the case of a uniform medium, where the equivalence is greatly simplified because the magnetic and electric vector potentials are not coupled.

According to the field–potential relations in Eq. (45), the field described by the magnetic vector potential $\mathbf{A}_\mu$ is equivalent to the field described by the electric vector potential $\mathbf{F}_\epsilon$ provided that

$$\mathcal{T}_\epsilon \mathbf{F}_\epsilon + \nabla \Psi = -\nabla \times \mathbf{A}_\mu \,, \tag{55}$$

or

$$\nabla \times \mathbf{F}_\epsilon = \mathcal{T}_\mu \mathbf{A}_\mu + \nabla \Phi \,. \tag{56}$$

In other words, the magnetic vector potential can be equivalently replaced by a properly chosen electric vector potential, and *vice versa*, so that the field remains unchanged.

If Eq. (55) is enforced, the $\mathbf{H}$ vector remains unchanged as indicated by the second expression in Eq. (45). The $\mathbf{E}$ vector also remains unchanged outside of the electrical sources $\mathbf{J}_e$. This statement is proved by taking the curl of both sides of Eq. (55), and taking into account that $\mathbf{A}_\mu$ satisfies the wave equation Eq. (40). The result is

$$\nabla \times \mathbf{F}_\epsilon = \mathcal{T}_\mu \mathbf{A}_\mu + \nabla \Phi - \mathcal{T}_\epsilon^{-1} \mathbf{J}_e \,. \tag{57}$$

Outside of the support of $\mathbf{J}_e$, Eq. (57) is identical with Eq. (56), and ensures unchanged $\mathbf{E}$ vector.

Similar argument applies if we assume that Eq. (56) holds: the $\mathbf{E}$ vector remains the same everywhere, while the $\mathbf{H}$ vector differs only at the location of nonzero magnetic currents (if any).

The above transformations between equivalent magnetic and electric vector potentials define yet another aspect of the nonuniqueness of the vector–potential field representation. It is utilized in numerical algorithms [13, 14] to scalarize the electromagnetic field, i.e., to represent it in terms of two scalar wave functions of space–time.

This nonuniqueness relates to the equivalence between electric and magnetic sources (see Theorem 4), or the type of nonradiating source discussed in Theorem 2. Assume that $\mathbf{A}'_\mu$ is due to $\mathbf{J}'_e = -\mathcal{T}_\epsilon \boldsymbol{\mathcal{E}}$, while $\mathbf{F}'_\epsilon$ is due to $\mathbf{J}'_m = -\nabla \times \boldsymbol{\mathcal{E}}$, and consider the simultaneous existence of both sources and their respective vector potentials. With these sources, Eq. (42) and Eq. (43) can be re-expressed as

$$\begin{aligned} -\nabla \times (-\mathcal{T}_\epsilon \mathbf{F}'_\epsilon - \nabla \Psi' + \nabla \times \mathbf{A}'_\mu) + \\ \mathcal{T}_\epsilon (-\mathcal{T}_\mu \mathbf{A}'_\mu - \nabla \Phi' - \nabla \times \mathbf{F}'_\epsilon - \boldsymbol{\mathcal{E}}) = 0\,, \end{aligned} \tag{58}$$

$$\begin{aligned} \nabla \times (-\mathcal{T}_\mu \mathbf{A}'_\mu - \nabla \Phi' - \nabla \times \mathbf{F}'_\epsilon - \boldsymbol{\mathcal{E}}) + \\ \mathcal{T}_\mu (-\mathcal{T}_\epsilon \mathbf{F}'_\epsilon - \nabla \Psi' + \nabla \times \mathbf{A}'_\mu) = 0\,, \end{aligned} \tag{59}$$

which leads to the Maxwell system,

$$\begin{aligned} -\nabla \times \mathbf{H} + \mathcal{T}_\epsilon (\mathbf{E} - \boldsymbol{\mathcal{E}}) = 0\,, \\ \nabla \times (\mathbf{E} - \boldsymbol{\mathcal{E}}) + \mathcal{T}_\mu \mathbf{H} = 0\,. \end{aligned} \tag{60}$$

The particular solution of this system is trivial, i.e., the simultaneous action of the sources $\mathbf{J}'_e = -\mathcal{T}_\epsilon \boldsymbol{\mathcal{E}}$ and $\mathbf{J}'_m = -\nabla \times \boldsymbol{\mathcal{E}}$ results in zero fields, with the exception that the $\mathbf{E}$ vector is nonzero at points where $\boldsymbol{\mathcal{E}}$ exists, $\mathbf{E} = \boldsymbol{\mathcal{E}}$, which is a result consistent with Theorem 2. Thus,

$$\mathcal{T}_\epsilon \mathbf{F}'_\epsilon + \nabla \Psi' = \nabla \times \mathbf{A}'_\mu\,, \tag{61}$$

and

$$-\nabla \times \mathbf{F}'_\epsilon = \mathcal{T}_\mu \mathbf{A}'_\mu + \nabla \Phi' + \boldsymbol{\mathcal{E}}\,. \tag{62}$$

In a linear medium, the total cancellation of the fields generated by $\mathbf{J}'_e = -\mathcal{T}_\epsilon \boldsymbol{\mathcal{E}}$ and $\mathbf{J}'_m = -\nabla \times \boldsymbol{\mathcal{E}}$ means that these fields are equal in magnitude but opposite in sign. Naturally, they can be made equivalent if the sign of one of the sources changes. For example, the field generated by $\mathbf{J}_e = -\mathcal{T}_\epsilon \boldsymbol{\mathcal{E}}$ alone, is the same as the field generated by $\mathbf{J}_m = \nabla \times \boldsymbol{\mathcal{E}}$, which is in accord with Theorem 4. Their respective vector potentials, $\mathbf{A}_\mu$ and $\mathbf{F}_\epsilon$, relate as

$$\mathcal{T}_\epsilon \mathbf{F}_\epsilon + \nabla \Psi = -\nabla \times \mathbf{A}_\mu\,, \tag{63}$$

and

$$\nabla \times \mathbf{F}_\epsilon = \mathcal{T}_\mu \mathbf{A}_\mu + \nabla \Phi + \boldsymbol{\mathcal{E}}\,. \tag{64}$$

Eq. (63) and Eq. (64), which relate the vector potentials due to equivalent sources, are identical with Eq. (55) and Eq. (57), which relate equivalent vector–potential field representations. We conclude that the nonuniqueness associated with field–invariant transformations between magnetic and electric vector potentials is due to the field invariance with respect to equivalent transformations between electric and magnetic sources.

We make a note that, as in the case of the gradient–type nonradiating source, the nonradiating combination of electric and magnetic currents does, in fact, generate two propagating vector potentials, one electric and one magnetic. Their net field, however, is zero outside of the source support.

D. Nonradiating sources for the vector potentials

To conclude this discussion, we point out that the vector potentials have their own set of nonradiating sources, i.e., sources whose vector potentials are zero outside of their support. These are derived according to Theorem 3. For example, the current density

$$-(\mathbf{J}_e^{nr})^A = \nabla^2 \boldsymbol{\mathcal{A}} - \mathcal{T}_{\mu\epsilon}\boldsymbol{\mathcal{A}}, \tag{65}$$

is a nonradiating vector–potential source in a uniform medium, see Eq. (40). Here, $\boldsymbol{\mathcal{A}}$ is an auxiliary vector, which is identical with the vector potential generated by $(\mathbf{J}_e^{nr})^A$.

As expected, sources exist, which are nonradiating for both the field vectors and the vector potentials. They belong to the type of sources described by Theorem 3, see Eq. (21) and Eq. (23). Certain limitations apply, however. It can be shown that a nonradiating source described by Eq. (21) or Eq. (23) is nonradiating for the respective vector potential as well only if it is also purely rotational, i.e., divergence–free. The nonradiating sources considered in the previous two sections do not belong to this group. As a result, although their field vectors vanish outside of their support, their vector potentials do not.

VI. CONCLUSION

We have derived the mathematical forms of three types of nonradiating electromagnetic sources in a nonuniform medium. Subsequently, we have used them to show that an infinite set of equivalent sources exists for a given original source: we derive the equivalence between electric and magnetic current densities, as well as recursive (second–order) equivalent source transformations. Higher–order recursive source transformations are possible provided that the respective derivatives of the original sources exist.

We consider the nonuniqueness of the electromagnetic field representation via vector potentials to be a consequence of the field invariance to the equivalent transformations of its sources. We examine the relationship between (i) the well–known gauge invariance of the field and its invariance to gradient–type current sources, and (ii) the field invariance to electric–magnetic source transformations and the respective magnetic–electric vector potential transformations. In these two cases, we show that the sources, which are nonradiating for the field vectors, are "radiating" for the vector potentials in the sense that the vector potentials they generate are not confined within their support.

Appendix A

We write the vector–potential wave equations for the case when only the nonradiating sources $\mathbf{J}_e^{nr} = \mathcal{T}_\epsilon \nabla P_e$ and $\mathbf{J}_m^{nr} = \mathcal{T}_\mu \nabla P_m$ exist:

$$\nabla^2 \mathbf{A}_\mu - \mathcal{T}_{\mu\epsilon}\mathbf{A}_\mu + (\nabla \mathcal{T}_\epsilon) \times \mathbf{F}_\epsilon + (\nabla \mathcal{T}_\epsilon)\, \Phi = -\mathcal{T}_\epsilon \nabla P_e, \tag{66}$$

$$\nabla^2 \mathbf{F}_\epsilon - \mathcal{T}_{\mu\epsilon}\mathbf{F}_\epsilon - (\nabla \mathcal{T}_\mu) \times \mathbf{A}_\mu + (\nabla \mathcal{T}_\mu)\, \Psi = -\mathcal{T}_\mu \nabla P_m. \tag{67}$$

The gauge is

$$\begin{aligned} -\mathcal{T}_\epsilon \Phi &= \nabla \cdot \mathbf{A}_\mu\,, \\ -\mathcal{T}_\mu \Psi &= \nabla \cdot \mathbf{F}_\epsilon\,. \end{aligned} \tag{68}$$

We re-write the vector–potential equations Eq. (66)–(67) as

$$\begin{aligned} -\nabla \times (-\mathcal{T}_\epsilon \mathbf{F}_\epsilon - \nabla \Psi + \nabla \times \mathbf{A}_\mu + \nabla P_m) + \\ \mathcal{T}_\epsilon(-\mathcal{T}_\mu \mathbf{A}_\mu - \nabla \Phi - \nabla \times \mathbf{F}_\epsilon + \nabla P_e) = 0\,, \end{aligned} \tag{69}$$

$$\begin{aligned} \nabla \times (-\mathcal{T}_\mu \mathbf{A}_\mu - \nabla \Phi - \nabla \times \mathbf{F}_\epsilon + \nabla P_e) + \\ \mathcal{T}_\mu(-\mathcal{T}_\epsilon \mathbf{F}_\epsilon - \nabla \Psi + \nabla \times \mathbf{A}_\mu + \nabla P_m) = 0\,, \end{aligned} \tag{70}$$

and we define auxiliary field vectors as—refer also to Eq. (45),

$$\begin{aligned} \mathbf{E}' &= -\mathcal{T}_\mu \mathbf{A}_\mu - \nabla \Phi - \nabla \times \mathbf{F}_\epsilon + \nabla P_e = \mathbf{E} + \nabla P_e\,, \\ \mathbf{H}' &= -\mathcal{T}_\epsilon \mathbf{F}_\epsilon - \nabla \Psi + \nabla \times \mathbf{A}_\mu + \nabla P_m = \mathbf{H} + \nabla P_m\,. \end{aligned} \tag{71}$$

It then follows from Eq. (69)–(70) that these vectors satisfy the system

$$\begin{aligned} \nabla \times \mathbf{H}' - \mathcal{T}_\epsilon \mathbf{E}' &= 0\,, \\ -\nabla \times \mathbf{E}' - \mathcal{T}_\mu \mathbf{H}' &= 0\,. \end{aligned} \tag{72}$$

As this is a homogeneous system of equations (of zero initial and boundary conditions), it has only a trivial particular solution. Therefore, see Eq. (71),

$$\mathbf{E} = -\nabla P_e \, , \, \mathbf{H} = -\nabla P_m \, . \tag{73}$$

This result is consistent with Theorem 1.

Appendix B

We seek the mathematical form of the vector potentials $\mathbf{A}_\mu$ and $\mathbf{F}_\epsilon$ in the case of gradient–type nonradiating sources, $\mathbf{J}_e^{nr} = \mathcal{T}_\epsilon \nabla P_e$ and $\mathbf{J}_m^{nr} = \mathcal{T}_\mu \nabla P_m$. We make use of Eq. (73) and the field–potential relations in Eq. (46), where $\mathcal{T}_\epsilon \mathbf{E}$ cancels $\mathbf{J}_e^{nr}$, and $\mathcal{T}_\mu \mathbf{H}$ cancels $\mathbf{J}_m^{nr}$. The result is a system of equations,

$$\begin{aligned} \nabla \times \nabla \times \mathbf{A}_\mu &= \nabla \times \mathcal{T}_\epsilon \mathbf{F}_\epsilon \, , \\ \nabla \times \nabla \times \mathbf{F}_\epsilon &= -\nabla \times \mathcal{T}_\mu \mathbf{A}_\mu \, , \end{aligned} \tag{74}$$

which relates $\mathbf{A}_\mu$ and $\mathbf{F}_\epsilon$ in space–time. These two vectors, however, are decoupled except at points of medium nonuniformities. The above can be true at any point of space–time for both uniform and nonuniform regions only if

$$\begin{aligned} \nabla \times \nabla \times \mathbf{A}_\mu &= \nabla \times \mathcal{T}_\epsilon \mathbf{F}_\epsilon = 0 \, , \\ \nabla \times \nabla \times \mathbf{F}_\epsilon &= -\nabla \times \mathcal{T}_\mu \mathbf{A}_\mu = 0 \, . \end{aligned} \tag{75}$$

We then conclude that

$$\mathcal{T}_\epsilon \mathbf{F}_\epsilon = \nabla \Lambda_\epsilon \, , \, \mathcal{T}_\mu \mathbf{A}_\mu = \nabla \Lambda_\mu \, , \tag{76}$$

where Λ_ϵ and Λ_μ are scalar functions.

To find the exact form of Λ_ϵ and Λ_μ, we substitute Eq. (76) in the field–potential relations of Eq. (45):

$$\begin{aligned} -\nabla P_e &= -\nabla \Lambda_\mu - \nabla \Phi - \nabla \times \mathbf{F}_\epsilon \, , \\ -\nabla P_m &= -\nabla \Lambda_\epsilon - \nabla \Psi + \nabla \times \mathbf{A}_\mu \, . \end{aligned} \tag{77}$$

Taking the divergence of both sides of each of the equations in Eq. (77) leaves us with two independent Laplace equations,

$$\nabla^2 (\Phi - P_e + \Lambda_\mu) = 0 \, , \tag{78}$$

and

$$\nabla^2(\Psi - P_m + \Lambda_\epsilon) = 0\,. \tag{79}$$

The case of zero boundary conditions leads to a trivial solution and a simple relation between the three scalar functions in each of the above equations:

$$\Lambda_\mu = P_e - \Phi\,,\ \Lambda_\epsilon = P_m - \Psi\,. \tag{80}$$

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