ELECTROMAGNETIC SOURCE EQUIVALENCE IN A NONUNIFORM MEDIUM

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Abstract: We study the electromagnetic field invariance to transformations of its sources in a nonuniform medium. We derive classes of nonradiating sources and then consider equivalent transformations of current densities. The scalarization (TE/TM decomposition) of the electromagnetic sources in an isotropic nonuniform medium is considered in the form of planar and linear distributions. We study the limitations of the linear distribution and show that it is applicable for a stratified medium. We focus on the field behavior inside the equivalent–source support since it may extend along lines or in planes throughout the computational volume. The concept is illustrated with numerical time–domain simulations.

1 INTRODUCTION

The equivalence of electromagnetic (EM) sources has been considered in the case of a uniform medium, e.g., [1]–[5]. Lindell showed that an infinite number of equivalent sources exists for a given electrical or magnetic current distribution in a uniform medium [3], and those may appear in the form of (i) series of localized sources, (ii) linear distributions, and (iii) planar distributions. The last two equivalent distributions also allow the TE/TM decomposition of the source with respect to a chosen direction $\hat{\mathbf{n}}$.

Recently, it has been shown that the construction of equivalent sources and the source TE/TM decomposition (also referred to as *source scalarization*) is possible in a nonuniform medium [7]. The analysis is carried out in the time domain. Specifically, it has been proven that in a nonuniform medium any set of electric–magnetic current density sources of arbitrary orientation can be reduced to an equivalent set of single–component ($\hat{\mathbf{n}}$-oriented) electric and magnetic current densities of planar distributions. Such transformation did not appear feasible for linear distributions.

In view of the importance of source equivalence in computational algorithms exploiting field TE/TM decomposition (field scalarization), we investigate the EM source equivalence and scalarization in a nonuniform medium. We derive expressions producing the exact field both inside and outside of the equivalent–source support. We discuss the linear–distribution scalarization in a stratified medium.

2 DEFINITIONS

The sources of propagating EM fields are time–varying electric and magnetic current densities, the latter being commonly treated as fictitious quantities.[1] Two sources are considered *equivalent* if they generate identical fields. In general, the fields have to be identical outside the volume of the original and equivalent source sets and their spatial derivatives;[2] however, they may differ inside their support.

A source generating an EM field confined within its support is called *nonradiating*. It is now obvious that two sources are equivalent if their difference is a nonradiating source of bound support. Their fields are equivalent outside of the support of the nonradiating difference source. Thus, the source equivalence requires clear understanding of the nonradiating EM sources.

3 NOTATIONS

First–order partial derivatives are denoted concisely by ∂_ξ, $\partial_\xi = \frac{\partial}{\partial \xi}$.

The operators $\mathcal{T}_\epsilon$ and $\mathcal{T}_\mu$ relate to the time–derivatives of the flux densities $\mathbf{D}$ and $\mathbf{B}$, respectively,

$$\partial_t \mathbf{D} = \mathcal{T}_\epsilon \mathbf{E}\,,\ \partial_t \mathbf{B} = \mathcal{T}_\mu \mathbf{H}\,. \qquad (1)$$

For example, in a dispersion–free medium, $\mathcal{T}_\epsilon = \boldsymbol{\epsilon}\partial_t + \boldsymbol{\sigma}_e$ and $\mathcal{T}_\mu = \boldsymbol{\mu}\partial_t + \boldsymbol{\sigma}_m$, where $\boldsymbol{\epsilon}$ is the permittivity, $\boldsymbol{\mu}$ is the permeability, $\boldsymbol{\sigma}_e$ and $\boldsymbol{\sigma}_m$ are the specific electric and magnetic conductivities, respectively. Eq. (1) are in effect the time–dependent constitutive relations and the involved constitutive parameters are tensors in the case of an anisotropic medium. We also use the second–order operator $\mathcal{T}_{\mu\epsilon} = \mathcal{T}_\mu \mathcal{T}_\epsilon$.

4 NONRADIATING CURRENT SOURCES IN A NONUNIFORM MEDIUM

Previous work [6, 2] shows that there are at least two classes of nonradiating current densities—those expressible in terms of the gradient of a scalar, and those which are functions of vectors. Here, we describe nonradiating EM sources in a nonuniform medium.[3]

[1] The lack of experimental evidence in support of the existence of magnetic monopoles leads to the understanding that magnetic currents are nonphysical fictitious EM sources.

[2] The volume where a source and its derivatives are nonzero is referred to as its *support*.

[3] We omit the proofs of all theorems for brevity.

Theorem 4.1 *Sources of bound support in the form of electric current density*

$$\mathbf{J}_e^{nr} = -\mathcal{T}_\epsilon \nabla P_e \tag{2}$$

and/or magnetic current density

$$\mathbf{J}_m^{nr} = -\mathcal{T}_\mu \nabla P_m \tag{3}$$

do not generate EM field outside of their own support. P_e and P_m can be any scalar functions of space–time whose first–order derivatives are well defined. The field they generate is localized at points of nonzero $\mathbf{J}_e^{nr}$, where

$$\mathbf{E} = \nabla P_e\,, \tag{4}$$

and nonzero $\mathbf{J}_m^{nr}$, where

$$\mathbf{H} = \nabla P_m\,. \tag{5}$$

Theorem 4.2 *Assume that the vector field $\mathbf{A}$ satisfies the equation*

$$L\mathbf{A} = \mathbf{G}\,, \tag{6}$$

where L is a linear operator, and $\mathbf{G}$ represents sources. If sources exist, which are expressible as

$$\mathbf{G}^{nr} = L\boldsymbol{\mathcal{A}}, \tag{7}$$

then these sources do not radiate beyond their own support. The vector $\boldsymbol{\mathcal{A}}$ is exactly the nonpropagating vector field generated by $\mathbf{G}^{nr}$, $\boldsymbol{\mathcal{A}} \equiv \mathbf{A}$, provided it satisfies the same boundary conditions as $\mathbf{A}$.

Having in mind the vector wave equations for the $\mathbf{E}$-field,

$$\nabla \times \mathcal{T}_\mu^{-1} \nabla \times \mathbf{E} + \mathcal{T}_\epsilon \mathbf{E} = -\mathbf{J}_e\,, \tag{8}$$

and for the $\mathbf{H}$-field,

$$\nabla \times \mathcal{T}_\epsilon^{-1} \nabla \times \mathbf{H} + \mathcal{T}_\mu \mathbf{H} = -\mathbf{J}_m\,, \tag{9}$$

we can conclude that (i) electrical sources of the type

$$\mathbf{J}_e^{nr} = \nabla \times \mathcal{T}_\mu^{-1} \nabla \times \boldsymbol{\mathcal{E}} + \mathcal{T}_\epsilon \boldsymbol{\mathcal{E}} \tag{10}$$

produce zero $\mathbf{E}$-field, and (ii) magnetic sources of the type

$$\mathbf{J}_m^{nr} = \nabla \times \mathcal{T}_\epsilon^{-1} \nabla \times \boldsymbol{\mathcal{H}} + \mathcal{T}_\mu \boldsymbol{\mathcal{H}} \tag{11}$$

produce zero $\mathbf{H}$-field. Here, $\mathcal{T}_\epsilon^{-1}$ and $\mathcal{T}_\mu^{-1}$ are the inverse operators of $\mathcal{T}_\epsilon$ and $\mathcal{T}_\mu$, respectively. If the nonradiating source produces zero $\mathbf{E}$-field everywhere—as in case (i) above—the $\mathbf{H}$-field may still be nonzero if there also exists a nonradiating magnetic source $\mathbf{J}_m^{nr}$, $\mathbf{H} = -\mathcal{T}_\mu^{-1}\mathbf{J}_m^{nr}$. Similarly, if the $\mathbf{H}$-field is zero everywhere—as in case (ii) above—the $\mathbf{E}$-field may still be nonzero if nonradiating electric sources are also present, $\mathbf{E} = -\mathcal{T}_\epsilon^{-1}\mathbf{J}_e^{nr}$.

5 EQUIVALENCE OF CURRENT SOURCES

Theorem 5.1 *The field generated by the magnetic current density $\mathbf{J}_m$ is identical to the field generated by the electric current density $\mathbf{J}_e$ outside the sources' support provided that (1)*

$$\nabla \times \mathcal{T}_\mu^{-1} \mathbf{J}_m = \mathbf{J}_e\,, \tag{12}$$

when, inside the source support, the $\mathbf{E}$-field is identical but the $\mathbf{H}$-field is different,

$$\mathbf{H}_e - \mathbf{H}_m = \mathcal{T}_\mu^{-1} \mathbf{J}_m \tag{13}$$

($\mathbf{H}_e$ is due to $\mathbf{J}_e$, and $\mathbf{H}_m$ is due to $\mathbf{J}_m$); or (2)

$$-\nabla \times \mathcal{T}_\epsilon^{-1} \mathbf{J}_e = \mathbf{J}_m\,, \tag{14}$$

when, inside the source support, the $\mathbf{H}$-field is identical, but the $\mathbf{E}$-field is not:

$$\mathbf{E}_m - \mathbf{E}_e = \mathcal{T}_\epsilon^{-1} \mathbf{J}_e\,. \tag{15}$$

Theorem 5.1 is a generalization of the equivalence of Mayes [1] for the case of a nonuniform medium.

Eq. (12) and Eq. (14) can be applied successively if the original source distribution has well defined higher–order derivatives; e.g., for the electric sources $\mathbf{J}_e^o = \mathcal{T}_\epsilon \boldsymbol{\mathcal{E}}^o$, an equivalent current $\mathbf{J}_e^e = \mathcal{T}_\epsilon \boldsymbol{\mathcal{E}}^e$,

$$\mathcal{T}_\epsilon \boldsymbol{\mathcal{E}}^e = -\nabla \times \mathcal{T}_\mu^{-1} \nabla \times \boldsymbol{\mathcal{E}}^o\,, \tag{16}$$

exists. The field $\mathbf{E}^e$ and $\mathbf{H}^e$ generated by $\mathbf{J}_e^e$ differ from the original field vectors $\mathbf{E}^o$ and $\mathbf{H}^o$ as

$$\mathbf{H}^e - \mathbf{H}^o = \mathcal{T}_\mu^{-1} \nabla \times \boldsymbol{\mathcal{E}}^o\,,\ \mathbf{E}^e - \mathbf{E}^o = \boldsymbol{\mathcal{E}}^o\,. \tag{17}$$

Similar (dual) transformations hold for magnetic sources, $\mathbf{J}_m^o = \mathcal{T}_\mu \boldsymbol{\mathcal{H}}^o$.

6 APPLICATIONS: AXIAL EQUIVALENT SOURCE DISTRIBUTIONS

The goal of source scalarization is to derive general expressions, which equivalently transform a set of current sources transversal to the distinguished axis $\hat{\mathbf{n}}$ into $\hat{\mathbf{n}}$-oriented sources. Thus, currents of any distribution and orientation—in general described by six spatial components—can be transformed

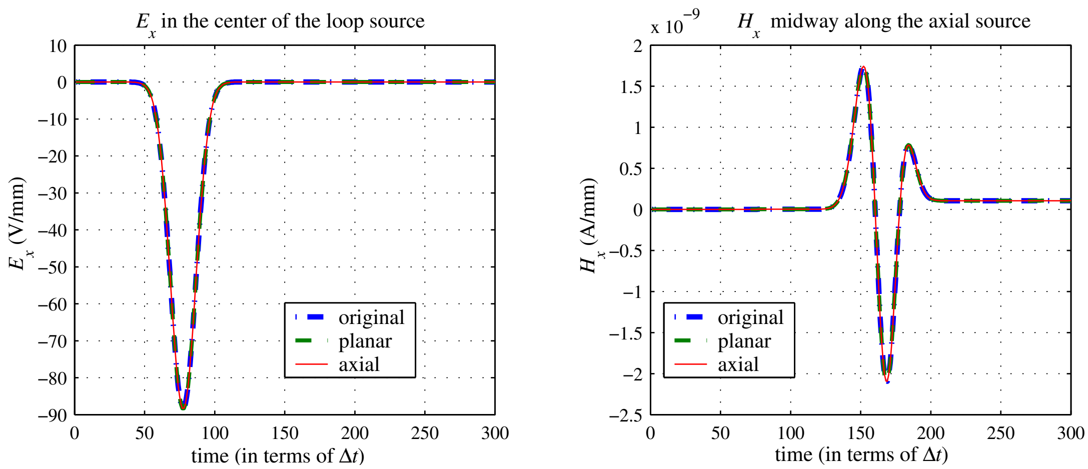

Fig. 1. Comparison of the E_x and H_x waveforms within the support of the axial equivalent source.

into single–component electric and magnetic sources, described in total by two spatial components. Georgieva *et al.* [7] derived scalarization formulas in the case of a nonuniform medium making use of a 3-D Helmholtz representation of the original source transversal components:

$$\mathbf{J}^o_{e\tau} = J^e_{en}\hat{\mathbf{n}} + \nabla \times \mathcal{H}_n \hat{\mathbf{n}} - \mathcal{T}_\epsilon \nabla P_e \,, \; \mathbf{J}^o_{m\tau} = J^e_{mn}\hat{\mathbf{n}} - \nabla \times \mathcal{E}_n \hat{\mathbf{n}} - \mathcal{T}_\mu \nabla P_m \,, \tag{18}$$

where $J^e_{en} = \partial_n P_e$ and $J^e_{mn} = \partial_n P_m$. The auxuliary functions $\mathcal{H}_n$, $\mathcal{E}_n$, P_e, and P_m are solutions to 2-D Poisson equations in the transverse planes (planar distributions) of nonzero original transverse sources [7]. They form the $\hat{\mathbf{n}}$-oriented equivalent sources,

$$J^e_{en} = \mathcal{T}_\epsilon(\mathcal{E}_n + \partial_n P_e) \,, \; J^e_{mn} = \mathcal{T}_\mu(\mathcal{H}_n + \partial_n P_m) \,. \tag{19}$$

Here, using the theory above, we show that the auxiliary functions can be found in the form of linear distributions along the $\hat{\mathbf{n}}$-axis (axial distributions) in the case of a stratified medium whose layers are transverse to $\hat{\mathbf{n}}$. A 1-D wave equation is solved along lines parallel to $\hat{\mathbf{n}}$ for each of them:

$$\mathcal{T}_{\mu\epsilon}(\mathcal{H}_n - \partial_n \mathcal{T}_\mu^{-1} \partial_n \mathcal{T}_\epsilon^{-1} \mathcal{H}_n) = g_{\mathcal{H}} \,, \; \mathcal{T}_{\mu\epsilon}(\mathcal{E}_n - \partial_n \mathcal{T}_\epsilon^{-1} \partial_n \mathcal{T}_\mu^{-1} \mathcal{E}_n) = g_{\mathcal{E}} \,, \tag{20}$$

$$\mathcal{T}_{\mu\epsilon}\Big((\partial_n P_m) - \partial_n \mathcal{T}_\mu^{-1} \partial_n \mathcal{T}_\epsilon^{-1} (\partial_n P_m)\Big) = g_{p_m} \,, \mathcal{T}_{\mu\epsilon}\Big((\partial_n P_e) - \partial_n \mathcal{T}_\epsilon^{-1} \partial_n \mathcal{T}_\mu^{-1} (\partial_n P_e)\Big) = g_{p_e} \,. \tag{21}$$

where $g_{\mathcal{H}} = -(\nabla_\tau \times \mathbf{J}^o_{e\tau}) \cdot \hat{\mathbf{n}}$, $g_{\mathcal{E}} = (\nabla_\tau \times \mathbf{J}^o_{m\tau}) \cdot \hat{\mathbf{n}}$, $g_{p_e} = -\partial_n(\mathcal{T}_\epsilon^{-1} \nabla_\tau \cdot \mathbf{J}^o_{e\tau})$, $g_{p_m} = -\partial_n(\mathcal{T}_\mu^{-1} \nabla_\tau \cdot \mathbf{J}^o_{m\tau})$.

The technique is particularly convenient in time–domain computations because it allows the use of explicit algorithms only (no matrix inversion is required). We demonstrate the concept through numerical time–domain simulations of the field radiated by an asymmetric loop [7] in a homogeneous and a stratified medium. We compare the fields generated by (i) the original magnetic loop whose axis is along $\hat{\mathbf{x}}$, (ii) the equivalent planar sources, and (iii) the equivalent axial sources. Fig. 1 shows (i) the waveform of the E_x field component recorded in the center of the magnetic loop—a point which is inside the support of all three sources, and (ii) the waveform of H_x at a point straight above (along $\hat{\mathbf{x}}$) the loop—a point which is inside the support of the axial equivalent source. The three fields are practically indistinguishable at all observation points. Excellent match (relative error below 10^{-5} for single–precision computations) is observed at all points of observation for all field components.

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