An Efficient Wavelet-Based Approach to Open Problems in Electromagnetics

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Outline

• Background
• Motivation and Theory
• Implementation - Examples
• Conclusion
Background: Classification of Well-Known Numerical Techniques for Full Wave EM Analysis

- Frequency Domain Methods
- Finite Element Methods
- Mode Matching Methods
- Moment Methods
- Time Domain Methods
- FDTD Methods
- TLM Methods
- IE Methods
Background: Classification of Well-Known Numerical Techniques for Full Wave EM Analysis

FREQUENCY-DOMAIN METHODS IN ELECTROMAGNETICS

DIFFERENTIAL EQUATION APPROACHES
- FDFD Techniques
- TLM Techniques
- FEM Techniques

INTEGRAL EQUATION APPROACHES
- Surface Integral Equation Techniques
- Volume Integral Equation Techniques
- Finite Integral Techniques
Motivation

Shortcomings of the known FD techniques

• larger problems handled at the expense of $O(N^3)$ increase in the computational load

• most improvements lead to increased complexity of the algorithms

• all require extensive changes in the available codes

Objective

Investigate the applicability of WT to general 3D EM problems for simpler and efficient solution
The integral equation (EFIE) approach

Define \( \vec{E}^s = -i\omega \vec{A} - \nabla \Phi \)

BC: \( \vec{n} \times (\vec{E}^i + \vec{E}^s) = \vec{0} \) on \( S \)

\( \nabla \cdot \vec{J} = -i\omega \sigma \) (continuity eq.)

Then

\[ -\vec{E}^i_{\tan} = \left( -i\omega \vec{A} - \nabla \Phi \right)_{\tan}, \text{ for } \vec{r} \text{ on } S \]

where

\( \vec{A}(\vec{r}) = \frac{\mu}{4\pi} \int_{S} \vec{J}(\vec{r}') \cdot \vec{G}_A(\vec{r}, \vec{r}') dS' \)

and \( \Phi(r) = \frac{1}{4\pi\varepsilon} \int_{S} \sigma(\vec{r}) G_q(\vec{r}, \vec{r}') dS' \)

\[ = \frac{\mu}{4\pi} \int_{S} \vec{J} \frac{e^{-ikR}}{R} dS' \text{ (in a free space)} \]

\[ = \frac{1}{4\pi\varepsilon} \int_{S} \sigma \frac{e^{-ikR}}{R} dS' \text{ (in a free space)} \]
The integral equation (EFIE) approach

Thus, the integro-differential equation for \( J \) (EFIE) is:

\[
\left\{ \frac{i \omega \mu}{4\pi} \int_S \mathbf{J}(\mathbf{r}') \cdot \mathbf{G}_A(\mathbf{r}, \mathbf{r}') dS' \right\} + \nabla_{\text{tan}} \left[ \frac{i}{4\pi \omega \varepsilon} \int_S \mathbf{J}(\mathbf{r}') \mathbf{G}_q(\mathbf{r}, \mathbf{r}') dS' \right] = \mathbf{E}_{\text{tan}}
\]

where

\( \mathbf{G}_A(\mathbf{r}, \mathbf{r}') \) is the dyadic Green’s function for the magnetic vector potential;

\( \mathbf{G}_q(\mathbf{r}, \mathbf{r}') \) is the scalar Green’s function for the charge scalar potential;

\( k = \omega \sqrt{\mu \varepsilon} = \frac{2\pi}{\lambda} \), where \( \lambda \) is the wavelength;

\( R = |\mathbf{r} - \mathbf{r}'| \) is distance between an observation point \( \mathbf{r} \) and a source point \( \mathbf{r}' \) on \( S \), both defined wrt global coordinate origin \( \theta \).
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Discretization – the Method of Moments

1. Choose suitable triangularization

2. Expand the unknown current

\[
\vec{J} = \sum_{n=1}^{N} I_n \vec{f}_n(\vec{r})
\]

where the basis functions are

\[
\vec{f}_n(\vec{r}) = \begin{cases} 
\frac{l_n}{2A^+_n} \rho^+_n, & \vec{r} \text{ in } T^+_n \\
\frac{l_n}{2A^-_n} \rho^-_n, & \vec{r} \text{ in } T^-_n \\
0, & \text{otherwise}
\end{cases}
\]

Discretization – the Method of Moments

3. Choose the symmetric inner product

\[ \langle \vec{f}, \vec{g} \rangle = \iint_S \vec{f} \cdot \vec{g} \, dS \]

and apply Galerkin’s method to render the problem into a matrix form

\[ ZI = V \]

where

\[ Z = [Z_{mn}] \in \mathbb{R}^{N \times N}, \quad I = [I_n] \in \mathbb{R}^{N \times 1}, \quad V = [V_m] \in \mathbb{R}^{N \times 1} \]
Discretization – the Method of Moments

The coefficient matrix elements are:

\[
Z_{mn} = l_m \left\{ i\omega \left[ \frac{\vec{A}_m^+ \cdot \vec{\rho}_m^+}{2} + \frac{\vec{A}_m^- \cdot \vec{\rho}_m^-}{2} \right] + \Phi_{mn}^- - \Phi_{mn}^+ \right\}
\]

The excitation vector components are:

\[
V_m = l_m \left[ \frac{\vec{E}_m^+ \cdot \vec{\rho}_m^+}{2} + \frac{\vec{E}_m^- \cdot \vec{\rho}_m^-}{2} \right]
\]

Therein,

\[
A_{mn}^\pm = \frac{\mu}{4\pi} \int \int_{S} \vec{f}_n(\vec{r}') \cdot \vec{G}_{A}(\vec{r}_m^\pm, \vec{r}') dS',
\]

\[
\Phi_{mn}^\pm = -\frac{1}{4\pi i\omega \varepsilon} \int \nabla' \cdot \vec{f}_n(\vec{r}') \vec{G}_{q}(\vec{r}_m^\pm, \vec{r}') dS',
\]
The Concept

Instead of looking for new wavelet basis functions, apply the discrete wavelet transform to the (already generated) MoM impedance matrix

- Solve for the unknown current coefficients in the spectral (wavelet) space
- Apply the inverse discrete wavelet transform to obtain the solution of the original problem
Solution - Comparison of few iterative methods

1. The Bi-Conjugate Gradient (BiCG) method for the split-into-real matrix equation with:
   - one- and three- diagonal preconditioners, and
   - preconditioner by Schulz’s fast iterative method
2. The Conjugate Gradient method for the normal equation (CG-NE) with:
   - one-diagonal preconditioner, and
   - incomplete Cholesky preconditioner;
3. Sarkar’s CG algorithm for complex-valued systems;
4. Schulz’s fast iterative method for the approximate inverse \( A_{m+1} = 2A_m - A_mA_Am \)
5. Reference (direct) solutions:
   - using the MATLAB built-in inverse matrix function (precompiled)
   - using the (not precompiled) inverse matrix (via LU decomposition)
   - using the FORTRAN built-in program LEQ2C
Advantages of the proposed method

- No change in the standard MoM code – simple implementation, the available software used directly
- Use of iterative methods with pre-sparsification of the coefficient matrix and preconditioning – computational speed increased
- MoM capacity enhanced with respect to large complex geometrical shapes as WT can handle irregular domains
Implementation - Algorithm

- Generate the complex MoM matrix $Z$ of self- and mutual impedances in $Z* I = V$

  where $I$ is the unknown current coefficients vector and $V$ is the known excitation vector

- Wavelet transform to $Z* I = V \rightarrow wZ* wI = wV$

- Sparsify $wZ$ using thresholds $\rightarrow swZ$

- Generate suitable preconditioner for $swZ$

- Solve $swZ* wI = wV$ by an iterative method

- Inverse-wavelet transform the solution vector $wI \rightarrow I$
Verification - Examples

1. Microstrip feeding line

2. Rectangular patch integrated antenna

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Current distribution along a microstrip line, \( N=512 \)

![Graph showing current distribution along a microstrip line, \( N=512 \). The x-axis represents normalized line length, and the y-axis represents normalized current. Peaks indicate the excitation end, and the open end is marked.]
Co-polar and cross-polar current distribution on a rectangular patch, N=512
Current distribution in the principal planes

- Rao, 60 elements
- Our result, 512 elements

Normalized patch length

Normalized current

Plane A-A'

Plane B-B'
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The MoM matrix for the patch, NE=1024

- $Z \in \mathbb{R}^{1024 \times 1024}$
- $wZ \in \mathbb{R}^{1024 \times 1024}$
- $swZ \in \mathbb{R}^{1024 \times 1024}$

WT – DAUB20

Nonzero pattern of the sparsified $wZ$
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The MoM matrix for the patch, NE=512

\[ Z \in \square \]

\[ wZ \in \square \]

\[ wZ \in \square \]

\[ s wZ \in \square \]

\[ Z \in \square \] (split)
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Comparison: $x = A\backslash b$ and WT_BiCG

<table>
<thead>
<tr>
<th>NE</th>
<th>MATLAB inverse not precompiled</th>
<th>tol=$10^{-6}$ BiCG – 1diag</th>
<th>$T_{\text{reduct}}$ [times]</th>
<th>tol=$10^{-3}$ BiCG – 1diag</th>
<th>$T_{\text{reduct}}$ [times]</th>
<th>tol=$10^{-2}$ BiCG – 1diag</th>
<th>$T_{\text{reduct}}$ [times]</th>
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</thead>
<tbody>
<tr>
<td>32</td>
<td>0.176</td>
<td>0.087</td>
<td>2.02</td>
<td>0.076</td>
<td>2.32</td>
<td>0.07</td>
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<td>64</td>
<td>0.736</td>
<td>0.258</td>
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<td>256</td>
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<td>512</td>
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<td>33.51</td>
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<tr>
<td>1024</td>
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<td>172.6</td>
<td>6.48</td>
<td>132.68</td>
<td>8.43</td>
<td>120.42</td>
<td>9.29</td>
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</tbody>
</table>

Sparsification 61%

Sparsification 72%
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Sparsification vs. error and run times

<table>
<thead>
<tr>
<th>NE</th>
<th>Run times (sec)</th>
<th>Error</th>
<th>$x_{dir} - x_{WT}$</th>
<th>max cut in $wZ$</th>
<th>$C_{comp} = \frac{NE^2}{nnz}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>Built-in inverse 0.006</td>
<td>0.0402</td>
<td>0.0532</td>
<td>55.12</td>
<td>2.229</td>
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<tr>
<td>64</td>
<td>Not-pre-compiled 0.176</td>
<td>0.0472</td>
<td>0.0526</td>
<td>59.01</td>
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<tr>
<td>128</td>
<td>BiCG – 1diag 0.258</td>
<td>0.0506</td>
<td>0.0500</td>
<td>50.83</td>
<td>2.034</td>
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<tr>
<td>256</td>
<td>$L_2$ norm 0.087</td>
<td>0.0506</td>
<td>0.0500</td>
<td>50.83</td>
<td>2.034</td>
</tr>
<tr>
<td>512</td>
<td>$L_\infty$ norm 0.0532</td>
<td>0.0506</td>
<td>0.0500</td>
<td>50.83</td>
<td>2.034</td>
</tr>
<tr>
<td>1024</td>
<td>$C_{comp} = \frac{NE^2}{nnz}$</td>
<td>0.0492</td>
<td>0.0494</td>
<td>71.61</td>
<td>3.522</td>
</tr>
</tbody>
</table>
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BiCG preconditioners - run times and number of iterations

<table>
<thead>
<tr>
<th>NE</th>
<th>% cut in wZ</th>
<th>Run times (sec)</th>
<th>Number of iterations</th>
</tr>
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<tr>
<td></td>
<td></td>
<td>1diag</td>
<td>3diag</td>
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<tr>
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<td>61.18</td>
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<td>1024</td>
<td>71.61</td>
<td>172.6</td>
<td>264.91</td>
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</table>

NE = 1024
## Comparison of the methods – time reduction

| Methods used, NE=512          | Run time | $\|error\|_{L_{\infty}}$ | $|error|_{L_2}$ | $T_{reduct}$ |
|-----------------------------|----------|-------------------------|----------------|-------------|
| MATLAB built-in inverse     | 8.48     | $<10^{-13}$             | $<10^{-13}$    | 19.42       |
| Not-precompiled inverse     | 145.2    | $<10^{-13}$             | $<10^{-13}$    | 1.00        |
| matrix                      |          |                         |                |             |
| BiCG without WT             | 113.65   | $3\times10^{-8}$        | $2.74\times10^{-8}$ | 1.27       |
| WT-BiCG – no truncation in  | 325.96   | $1.2\times10^{-12}$     | $1.16\times10^{-12}$ | 0.45       |
| wZ                          |          |                         |                |             |
| WT-BiCG with 61% truncation | 18.59    | $1.92\times10^{-2}$     | $1.89\times10^{-2}$ | 7.81       |
| WT-BiCG with 30% truncation | 36.17    | $1.57\times10^{-2}$     | $1.28\times10^{-2}$ | 4.02       |
| LEQ2C – in FORTRAN          | -        | $3.2\times10^{-5}$      | $2.2\times10^{-5}$ | -           |
Conclusion

- Efficient implementation of wavelet transform to solve 3D EM field problems proposed.

- Significant run time reduction for large problems (N>1000) using BiCG method in comparison with the direct methods.

- Sparsification limited by conceptual considerations re. the underlying physics of the EM field problem.

- Negligible programming overhead – no change in the MoM code required.

- Few iterative methods with suitable preconditioners compared – BiCG with 1-diagonal preconditioner offers the shortest run times.
Further developments

- In MoM use combination of rectangular and triangular subdomains to reduce the number of unknowns.

- Search for better threshold algorithms reflecting the EM nature of the problem and allowing for bigger sparsification.

- Testing the algorithm on larger problems – expected increased efficiency due to the bigger compression factor.